ON THE MAXIMAL NUMBER OF SOLUTIONS OF A PROBLEM IN LINEAR INEQUALITIES

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ABSTRACT

The problem $y = Ax + c$, $x \geqq 0$, $y \geqq 0$, $(x, y) = 0$ is considered, where the square real matrix A and the real vector c are the data and a solution is a pair of vectors x, y. Under certain conditions on the matrix A there exists a solution for every vector c, but it cannot be unique for every c. We prove that under these conditions the maximal number of solutions is $2^n - 1$.

1. Introduction

In a recent paper [1], A. W. Ingleton has studied the following problem in the theory of linear inequalities: Given a square real matrix $A = (a_{ij})_{i,j=1}^n$ and an n-dimensional real vector $c = (c_i)_{i=1}^n$, under what conditions on the matrix A do there exist solutions of the system:

$$(*) \qquad y = Ax + c, \ x \geqq 0, \ y \geqq 0, \ (x, y) = 0$$

for every vector c? and when is the solution of the system (*) unique? Clearly, by a solution we mean a pair of vectors x and y which satisfy (*). (x, y) stands for the inner product of the two vectors in the n-dimensional vector space V.

Finding solutions of the system (*) is known in mathematical programming as the linear complementarity problem, see [2].

In [1], Ingleton has proved that under certain conditions there exists a solution of (*) for every vector c, but it cannot be unique for every c, and conjectured that under these "certain conditions", the maximal number of solutions of (*) is $2^n - 1$.

The purpose of the present note is to answer in part this conjecture.

2. Definitions and notations

Let V be a real vector space of finite dimension $n > 0$ endowed with an inner product denoted by (x, y) and $e_1, e_2, \cdots, e_n$ an orthonormal basis of V (which is assumed to remain fixed throughout except where otherwise stated). A vector

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$x \in V$ will be called non-negative, written $x \geqq 0$ if $(x, e_i) \geqq 0$ for all $i = 1, 2, \cdots, n$; semi-positive, written $x \geq 0$ if $x \geqq 0$ and $x \neq 0$; and positive, written $x > 0$ if $(x, e_i) > 0$ for all $i = 1, 2, \cdots, n$.

The range of a linear transformation $A: V \to V$ will be denoted by $R(A)$, its rank by ρA, and its adjoint by A^* (i.e. $(Ax, y) = (x, A^* y)$ for all $x, y \in V$).

Given any subset I of $\{1, 2, \cdots, n\}$ we denote by

$|I|$ the number of elements in I,
V_I the subspace spanned by $\{e_i : i \in I\}$,
A_I the restriction of A to V_I,
P_I the orthogonal projection of V onto V_I,
A_{II} the linear transformation $P_I A_I : V_I \to V_I$,
x_I the vector $P_I x$ $(x \in V)$.

(When $I = \phi$, V_ϕ is the zero vector space, and for any $A, A_{\phi\phi}$ is the unique map $V_\phi \to V_\phi$ which is invertible and has determinant unity).

DEFINITION. The linear transformation A will be called adequate (with respect to a given basis) if for every subset I of $\{1, 2, \cdots, n\}$,

1) $\det A_{II} \geqq 0$
2) $\det A_{II} = 0 \Rightarrow \rho A_I < |I|$
3) $\det A_{II} = 0 \Rightarrow \rho A_I^* < |I|$

The matrix of an adequate linear transformation A (with respect to a given basis) has therefore the following properties:

a) all principal minors of A are non-negative,

b) if some principal minor of A vanishes then the corresponding rows of A are linearly dependent and so are the corresponding columns.

3. Sufficient conditions for the existence and uniqueness of solutions

The following theorems, corollary and lemma were proved by Ingleton in [1]:

THEOREM 1. *If $A: V \to V$ is an adequate linear transformation then for every $c \in R(A)$ there is exactly one $y \in V$ such that, for some x, $y = Ax + c$, $x \geqq 0$, $y \geqq 0$, $(x, y) = 0$; if A is invertible then x is also unique.*

THEOREM 2. *If $A: V \to V$ is a linear transformation such that for every $c \in V$ there is at most one $x \in V$ such that $x \geqq 0$, $Ax + c \geqq 0$, $(x, Ax + c) = 0$, then A is adequate and invertible.*

THEOREM 3. *Let the linear transformation* $A: V \to V$ *be such that* A_{JJ} *is invertible for every subset* J *of* $\{1,2,\cdots,n\}$. *Suppose that for some* $c \in V$ *there is exactly one pair of vectors* x, y *satisfying*:

$$(*) \qquad y = Ax + c, \ x \geqq 0, \ y \geqq 0, \ (x,y) = 0$$

and that $x + y > 0$. *Then at least one solution of* (*) *exists for every* $c \in V$.

From this theorem follows immediately:

COROLLARY. *If* A_{JJ} *is invertible for every* J, *and if for some* $c > 0$ *the only solution of* (*) *is* $x = 0$, $y = c$, *then at least one solution of* (*) *exists for every* $c \in V$.

LEMMA. *Each of the following properties of a linear transformation* A *implies the succeeding ones, and the last two are equivalent.*

a) $(Ae_i, e_j) \geqq 0$ *for all* i,j *and* $(Ae_i, e_i) > 0$ *for all* i.
b) *For every* J, *there exists* $p \in V_J$ *such that* $p \geqq 0$, $A_{JJ}^* p > 0$.
c) *For every* J, *and every* $u \in V_J$

$$u \geqq 0, \ A_{JJ} u \leqq 0 \Rightarrow u = 0.$$

d) *For any* $c \geqq 0$, *the sytem* (*) *is satisfied only if* $x = 0$.

We add here that in this Lemma (c) also implies (b).

PROOF. Assume (c) and take its transpose which reads

$$u' \geqq 0, \ u' A_{JJ}^* \leqq 0 \Rightarrow u' = 0$$

then by a well known theorem [3; Theorem 2.10] (b) follows.

The following theorem is an immediate consequence of the last lemma and corollary, [1]:

THEOREM 4. *If the linear transformation* $A: V \to V$ *is such that* A_{JJ} *is invertible for every subset* J *of* $\{1,2,\cdots,n\}$, *and if moreover* A *satisfies any of the conditions* (a) *to* (d) *of the Lemma, then at least one solution of the system* (*) *exists for every* $c \in V$.

4. Ingleton's conjecture

It follows from Theorem 2 that the solution of the system (*) cannot be unique for every c in the following two cases, *Case I. A satisfies the hypotheses of Theorem* 4 *without being adequate.*

Case II. A satisfies the hypotheses of Theorem 3 *without being adequate.*

Ingleton conjectured ([1], p. 534, §4.5) that in both cases the maximal number of solutions of the system (*) in n dimensions is $2^n - 1$.

Case I. The proof of the conjecture consists of the following three steps:

Step A. If A has the property that A_{JJ} is invertible for every subset J of $\{1,2,\cdots,n\}$ then the number 2^n is an upper bound for the number of solutions which the system (*) can have.

Step B. If A satisfies the hypotheses of Theorem 4 then the number $2^n - 1$ is an upper bound for the number of solutions which the system (*) can have.

Step C. There exists a matrix A and a vector c for which the system (*) has exactly $2^n - 1$ solutions.

PROOF.

Step A. The requirement $(x, y) = 0$ restricts the number of solutions that the system (*) can have.

Indeed if for some i $(1 \leqq i \leqq n)$ $x_i \neq 0$ then y_i must be zero and if for some j $(1 \leqq j \leqq n)$ $y_j \neq 0$ then x_j must be zero. Now, if a vector x has some p components equal to zero $(1 \leqq p \leqq n)$ then the pair x, y which forms a solution of the system (*) is uniquely determined, i.e. there does not exist another pair x', y' which is a solution of (*) and in which x' has the same p components equal to zero as x has.

Let us prove this assertion. Without loss of generality assume that x is a vector whose first p components $x_1, x_2, \cdots, x_p$ are zero and whose last $(n - p)$ components $x_{p+1}, \cdots, x_n$ are different from zero.

Then $(x, y) = 0$ assures that the corresponding vector y has its last $(n - p)$ components equal to zero.

The last $(n - p)$ equations of the system $y = Ax + c$ will then be:

$$(1) \qquad \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \begin{bmatrix} & & \\ & A_{JJ} & \\ & & \end{bmatrix} \begin{bmatrix} x_{p+1} \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} c_{p+1} \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

where A_{JJ} is the lower right hand principal minor of order $(n - p)$ of A.

The system (1) is a system of linear equations with a non-vanishing determinant and therefore $x_{p+1}, x_{p+2}, \cdots, x_n$ are uniquely determined.

Then for $1 \leqq i \leqq p$ the first components of y are uniquely determined by:

$$y_i = \sum_{k=p+1}^{n} a_{ik}x_k + c_i, \qquad 1 \leqq i \leqq p \tag{2}$$

Now a bound for the number of solutions of the system (*) is the number of all possible solutions under the restriction $(x, y) = 0$. Clearly, this bound is 2^n.

Step B. We claim that there does not exist a vector c for which the system (*) has 2^n solutions.

For if $c \geqq 0$ then (*) has only one solution according to (d) of the lemma of the previous section.

If $c \ngeqq 0$ then the solution $x = 0$, $y = c$ is impossible since it means that $y \ngeqq 0$, contrary to (*).

Therefore, if A satisfies the hypotheses of Theorem 4 the number $2^n - 1$ is an upper bound for the number of solutions of the system (*).

Step C. Take $A = (a_{ij})_{i,j=1}^n$ with $a_{ii} = 1$ for all i and $a_{ij} = 2$ for all i,j, $i \neq j$; and take $c = (c_i)_{i=1}^n$ with $c_i = -1$ for all i. We will prove that for these A and c: (i) A satisfies the hypotheses of Theorem 4 (and therefore also those of Theorem 3), (ii) A is not adequate, and (iii) the system (*) has exactly $2^n - 1$ solutions.

i) Denote the determinant of a principal submatrix of order J of the matrix A by $A(J)$, then

$$A(J) = (-1)^{J-1}(2J-1) \neq 0 \tag{4}$$

which means that A_{JJ} is invertible for every J.

A also satisfies (a) of the lemma of Section 3.

ii) A is not adequate since all its principal minors of even order are negative.

iii) We have to show that every pair of vectors of the form

$$x = (x_1, x_2, \cdots, x_p, 0, \cdots, 0),\ y = (0, \cdots, 0, y_{p+1}, y_{p+2}, \cdots, y_n),$$

which is a solution of $y = Ax + c$ satisfies $x \geqq 0$, $y \geqq 0$.

The system $y = Ax + c$ splits into two systems,

$$0 = A_{JJ}x_J + c_J \tag{5}$$

with $|J| = p$, and

(6) $$y_i = 2\left[\sum_{k=1}^{p} x_k\right] - 1, \qquad i = p+1, p+2, \cdots, n.$$

Solving the system (5) we get

$$x_i = \frac{1}{2p-1} \geqq 0, \qquad i = 1, \cdots, p.$$

and the system (6) then gives

$$y_j = \frac{1}{2p-1} \geqq 0, \qquad j = p+1, \cdots, n.$$

Q.E.D.

REMARKS.

1) One can easily see that the maximal number of solutions for (*) under the discussed conditions is obtained not only for the data given in the proof of Step C, but also for many other data as for example: $A = (a_{ij})_{i,j=1}^{n}$ with $a_{ii} = 1$ for all i and $a_{ij} = k$ for all i,j $i \neq j$ and the vector $c = (c_i)_{i=1}^{n}$ with $c_i = -p$ for all i, where k and p are suitable natural numbers.

2) For the special case $n = 2$ the following is true:
For *every* matrix $A = (a_{ij})_{i,j=1}^{2}$ which satisfies

i) $\det A_{JJ} \neq 0$ for every J,

ii) $a_{ij} \geqq 0$ for all i,j and $a_{ii} > 0$ for all i,

iii) A is not adequate,

a vector c can be constructed so that the system (*) will attain its maximal number of solutions (i.e. three). The proof is a straightforward one and appears in [4].

Case II. Steps A and C of the proof of Case I hold also here but we did not manage to reduce the upper bound for the number of solutions of (*) to $2^n - 1$.

Using some results of Murty [5] it is possible to settle Case II partially.

DEFINITION [5; §2.16]. The vector c is called *nondegenerate with respect to* A if and only if for all solutions of $y = Ax + c$ at most n of the $2n$ variables $\{x_i, y_i\}$ are zero.

THEOREM 5 [5; Th.6.2]. *If A satisfies $\det A_{JJ} \neq 0$ for every J then the number of solutions of* (*) *has the same parity* (*odd or even*) *for all c nondegenerate with respect to A.*

Now, A satisfies the hypotheses of Theorem 3 means that for some c nondegenerate with respect to A there is a unique solution of the system (*). Then, according to Theorem 5, the number of solutions of (*) must be odd for every c nondegenerate with respect to A.

Thus for Case II the maximal number of solutions of (*) is $2^n - 1$ when the maximum is taken over the data $\{A, c\}$ where A is of case II and c is nondegenerate with respect to A.

5. Discussion

Consider the class of matrices A which satisfy:

i) $\det A_{JJ} \neq 0$ for every J, and

ii) the system (*) has a solution for every $c \in R^n$,

and denote this class by $\boldsymbol{G}$.

An invertible matrix is adequate if and only if it is a $\boldsymbol{P}$-matrix (i.e. has $\det A_{JJ} > 0$ for every J), see [1], [2]. Therefore, within the class $\boldsymbol{G}$ the adequate matrices are exactly all the $\boldsymbol{P}$-matrices.

For a matrix A which is in $\boldsymbol{G}$ but not a $\boldsymbol{P}$-matrix, (*) has a solution for every c but again according to Theorem 2 the solution of (*) cannot be unique for every c. This leads to the following

QUESTION. Is it true that for every matrix A in $\boldsymbol{G}$ which is not a $\boldsymbol{P}$-matrix the maximal number of solutions of the system (*) is $2^n - 1$?

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